SOME FUNDAMENTAL THEOREMS IN BANACH SPACES AND HILBERT SPACES

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- Functional analysis is the branch of mathematics, concerned with the study of certain topological-algebaric and geometric sttuctures and techniques by which knowledge of these structures can be applied to analytic problems. Functional analysis play crucial role in the applied sciences as well as in mathematics. Functional analysis has its applications in every branch of mathematics like in mathematical finance, differential equations, computer science, probability theory, quantum mechanics, quantum Physics etc.
- The early development of Functional analysis is due to Polish mathematician Stefan Banach and his group around 1920's. Hilbert spaces are special Banach spaces. Historically they are older than general normed spaces. A Hilbert space is the abstraction of the finite-dimensional Euclidean spaces and retains many features of Euclidean spaces, a central concept being orthogonality. In fact, inner product spaces are probably the most natural generalization of Euclidean space. The whole theory was initiated by the work of D. Hilbert (1912) and E. Schmidt (1908). The Banach spaces and Hilbert spaces are more important spaces that we met in daily life and upon which every scientist can rely throughout his or her career.

The aim of this course is to introduce the student to the key ideas of functional analysis. It should be noted that only scratch the surface of this vast area in this course. We discuss normed linear spaces, Hilbert spaces, bounded linear operators and the fundamental results in functional analysis such as Baires category theorem, the Hahn-Banach theorem, the uniform boundedness principle, the open mapping theorem, the closed graph theorem and the Riesz's Representation theorem.

Definition

Let X be a vector space over either the scalar field $\mathbb{R}$ of real numbers or the scalar field $\mathbb{C}$ of complex numbers.Suppose we have a function $||.|| : X \to [0, \infty)$ such that

(1) $||x|| = 0$ if and only if $x = 0$;

(2) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$; and

(3) $||\alpha x|| = |\alpha|\ ||x||$ for all scalars α and vector x.

We call $(X, ||.||)$ a normed linear space.

Property (2) is called the triangle inequality and property (3) is referred to as homogeneity. The reverse triangle inequality,

$$||x + y|| \geq ||x|| - ||y||.$$

Now we state the Hahn Banach Theorem for real linear space

Theorem

(Hahn Banach Theorem) *Let X be a real linear space and let p be a sublinear functionals on X. If f is a real linear functional which is defined on a subspace Z on X satisfying*

$$f(x) \leq p(x)$$

for all $x \in M$, then there exists a real linear functional f on X such that

$$\widetilde{f}|_Z = f \text{ and } \widetilde{f}(x) \leq p(x), \;\; \forall \;\; x \in X.$$

Hahn-Banach Theorem for normed spaces

Hahn-Banach Theorem for normed spaces

Theorem

Let X be a normed space over the field $\mathbb{K}$ and Z be a subspace of X. Then, for every bounded linear functional f on Z, there exist a bounded linear functional $\widetilde{f}$ on X such that

$$\widetilde{f}|_Z = f \text{ and } ||\widetilde{f}||_X = ||f||_Z,$$

where $||\widetilde{f}||_X = \sup_{x\in X,\ \|x\|=1} |\widetilde{f}(x)|$, *and* $||f||_Z = \sup_{x\in X,\ \|x\|=1} |f(x)|$.

Some applications of Hahn Banach Theorem

Corollary

Let X be a normed space and let $0 \neq x_0 \in X$. Then there exists $\widetilde{f} \in X'$ with $||\widetilde{f}|| = 1$ and $\widetilde{f}(x_0) = ||x_0||$.

Corollary

Let X be a normed space over the field $\mathbb{K}$ and x_1 and x_2 be distinct points of X. Then there exists a $\widetilde{f} \in X'$ such that

$$\widetilde{f}(x_1) \neq \widetilde{f}(x_2).$$

The next fundamental theorem is Open Mapping Theorem

Corollary

Let X be a normed space over the field $\mathbb{K}$ and x_1 and x_2 be distinct points of X. Then there exists a $\widetilde{f} \in X'$ such that

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The next fundamental theorem is Open Mapping Theorem

Theorem

(Open Mapping Theorem) *Let X and Y be Banach spaces over the field $\mathbb{K}$ and $T : X \to Y$ be a bounded linear operator. Then T is an open mapping.*

Conditions of the completeness in open mapping is essential

Example

The Banach space $X = (C[0,1], ||.||_\infty)$ and the normed space $Y = (C[0,1], ||.||_1)$ and the identity operator $I : X \to Y$. Then clearly I is continuous (hence bounded). Since $||Ix||_1 = ||x||_1 \leq ||x||_\infty$. Also, I is bijective so we can consider the inverse mapping $I^{-1} : Y \to X$ which is again linear. But I^{-1} is not continuous.

The following theorem is an intermediate consequence of open mapping theorem

Theorem

(Inverse Mapping Theorem) *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator if T is bijective, then T^{-1} is a bounded linear operator.*

Definition

Let X and Y be normed spaces and $T : D(T) \to Y$ a linear operator with domain $D(T) \subset X$. Then T is called a closed linear operator if its graph $G(T) = \{(x, y) : x \in D(T), y = Tx\}$ is closed in the normed space $X \times Y$, where the two algebraic operations of a vector space in $X \times Y$ are defined as usual, i.e.,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$\alpha(x, y) = (\alpha x, \alpha y), \textit{where } \alpha \textit{ is a scalar}.$$

The norm on $X \times Y$ is defined by

$$||(x, y)|| = ||x|| + ||y||.$$

Theorem

Let X and Y be normed spaces and $T : D(T) \to Y$ be a linear operator where $D(T) \subset X$. Then T is closed if and only if it has the following properties if $x_n \to x$, where $x \in D(T)$ and $Tx_n \to y$. Then $x \in D(T)$ and $Tx = y$.

The following theorem is known as closed graph theorem

Theorem

Let X and Y be Banach spaces and $T : D(T) \to Y$ be a closed linear operator, where $D(T) \subset X$.Then if D(T) is closed in X, the operator T is bounded.

In general, we observed that the closedness need not imply boundedness and boundedness need not imply closedness

Example

Consider the Banach space $(C[0,1], ||.||_\infty)$ and the normed space $(C'[0,1], ||.||_\infty)$. Define the mapping $T : C'[0,1] \to C[0,1]$ by $(Tx)(t) = x'(t),\ x \in C'[0,1], t \in [0,1]$.
Then
(i) T is a linear operator.
(ii) T is not bounded.
(iii) T is closed.

Theorem

*(***The Banach Steinhaus Theorem)** *Consider a Banach space X and a normed linear space Y. If $A \subset B(X,Y)$ is such that $sup\{||Tx||_Y | T \in A\} < \infty$ for each $x \in X$, then $sup\{||T||_{B(X,Y)} | T \in A\} < \infty$.*

Definition

Let X be a linear space over the complex filed $\mathbb{C}$. An inner product on X is a function $\langle\ \rangle : X \times X \to \mathbb{C}$ which satisfies the following conditions:
(1) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$; (Linearity in the first variable)
(2) $\overline{\langle x, y\rangle} = \langle y, x\rangle$ where the bar denotes the complex conjugate ;(symmetry)
(3) $\langle x, x\rangle \geq 0, \langle x, x\rangle = 0$ if and only if x $= 0$. (positive definiteness)

A complex inner product space X is a linear space over $\mathbb{C}$ with an inner product defined on it.

Definition

A complete inner product space is called a Hilbert space.

Next, we give the example of a Banach space which is not a Hilbert space

Example

The space ℓ_p with $p \neq 2$ is not an inner product space and hence not a Hilbert space.

Reisz's Reprsentation Theorem

Theorem

Every bounded linear functionals f on a Hilbert space H can be represented in terms of the inner product, namely

$$f(x) = \langle x, z \rangle,$$

where z depends on f is uniquely determined by f and has norm

$$||f|| = ||z||.$$

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