Wavelets: Theory and Applications

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Contents I

The simplest method to represent a function is to use point sampling. In the one dimensional case a real function of one-variable $f : I \subset \mathbb{R} \to \mathbb{R}$ is discretized by taking a partition $t_1 < t_2 < \cdots < t_n$ of the domain interval I. The representation is given by the vector

$$f_n = \big(f(t_1), f(t_2), \ldots, f(t_n)\big) \in \mathbb{R}^n$$

.

In this way, the space of real functions defined on the interval I is represented by the Euclidean space $\mathbb{R}^n$.

Fourier series

A Fourier series may be defined as an expansion of a 2π-periodic function f in series by sines and cosines and is given by

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

where

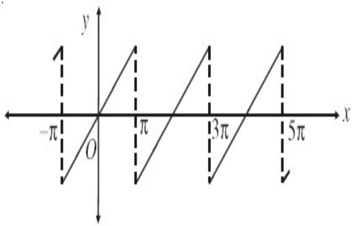
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \qquad k = 0, 1, \ldots$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \qquad k = 0, 1, \ldots$$

Example 1

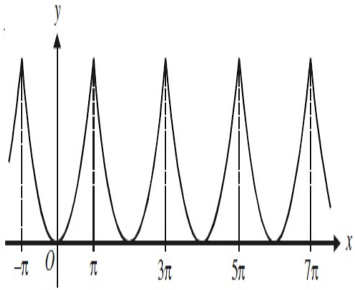
Let $f(x) = x, x \in [-\pi, \pi]$. Then the Fourier series of the given function is given by



$$f(x) = 2\left[\sin x - \sin\frac{2x}{2} + \sin\frac{3x}{3} - \cdots\right]$$

Example 2

Let $f(x) = x^2, x \in [-\pi, \pi]$. Then the Fourier series of the given function is given by



$$f(x) = 2\left[\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx\right]$$

Fourier transform

The Fourier transform of any function $f(x) \in L^2(\mathbb{R})$ is given by

$$\mathcal{F}(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt$$

and its Inverse Fourier transform is

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{i\xi t} d\xi.$$

Example(Characteristic function)

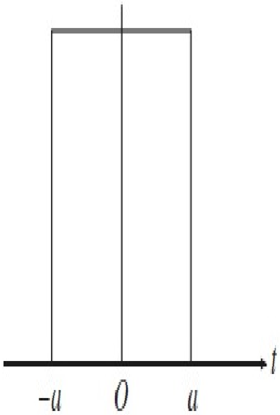
Consider the function

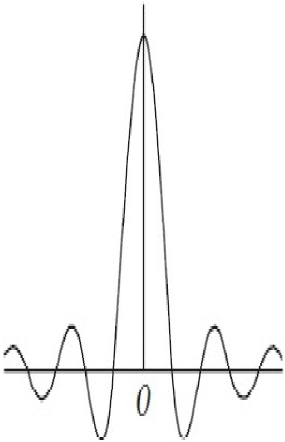
$$f(t) = \begin{cases} 1 & |t| \leq a \\ 0 & |t| > a \end{cases}, \quad a > 0.$$

Then the Fourier transform $\hat{f}(\xi)$ of $f(t)$ is given by

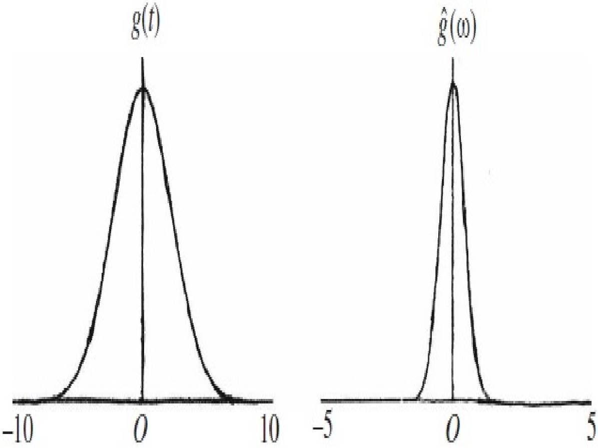
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt = \frac{2 \sin a\xi}{\xi}.$$

Characteristic function and its Fourier transform





Gaussian function and its Fourier transform



Let $f, g \in L^1(\mathbb{R})$ and α, β be any two complex constants. Then the following properties of Fourier transform hold:

1. Linearity:

$$\mathcal{F}[\alpha f(t) + \beta g(t)] = \mathcal{F}[\alpha f(t)] + \mathcal{F}[\beta g(t)] = \alpha \hat{f}(\xi) + \beta \hat{g}(\xi).$$

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2. Time shift $\rightarrow$ frequency modulation:

$$\mathcal{F}[T_a f(t)] = \mathcal{F}[f(t-a)] = e^{-ia\xi} \hat{f}(\xi) = M_{-a} \hat{f}(\xi).$$

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3. Scaling: For $a \neq 0$,

$$\mathcal{F}\left[D_{\frac{1}{a}} f(t)\right] = D_a \hat{f}(\xi) = \frac{1}{\sqrt{a}} \hat{f}\left(\frac{\xi}{a}\right).$$

Properties Contd...

4. Exponential modulation $\rightarrow$ frequency shift:

$$\mathcal{F}[M_a f(t)] = \mathcal{F}[e^{iat} f(t)] = \hat{f}(\xi - a) = T_a \hat{f}(\xi).$$

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5. Change of roof:

$$\int_{-\infty}^{\infty} \hat{f}(t) g(t) dt = \int_{-\infty}^{\infty} f(t) \hat{g}(t) dt.$$

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- It does not give any information regarding to measure the position and momentum of particle simultaneously at a particular point.

A Walk in The Physical Universe

Our purpose is to obtain a transform that enables us to perform a local computation of the frequency density. The inspiration for this transform is to analyze the audio analysis performed by our auditory system. Consider for this an audio signal represented by a real function f of one variable (time).

A Walk in The Physical Universe

Our purpose is to obtain a transform that enables us to perform a local computation of the frequency density. The inspiration for this transform is to analyze the audio analysis performed by our auditory system. Consider for this an audio signal represented by a real function f of one variable (time).

- **Real time analysis:** The audio information we receive occurs simultaneously on time and frequency. This means that the signal f is transformed by the auditory system in a signal $\tilde{f}(t, \omega)$ that depends on the time and the frequency.

- **Future sounds are not analyzed:** This means that only values of $f(t)$ for $t \leq t_1$ can be analyzed when computing the transform $\tilde{f}(t, \omega)$.

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- **The auditory system has finite memory:** That is, sounds that we have heard some time ago do not influence the sounds that we hear in a certain instant of time. This means that there exists a real number $t_0 > 0$ such that the computation of the transform $\tilde{f}(t, \omega)$ depends only on the values of t on the interval $[t - t_0, t]$.

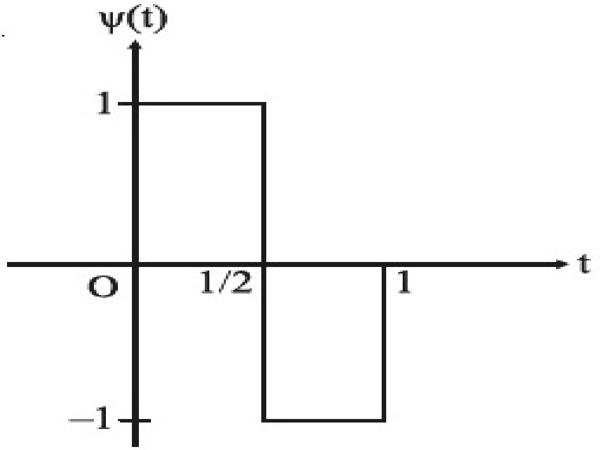
Mathematically, the last two properties show that the modulating function used to detect frequencies in the computation of the transform $\tilde{f}(t, \omega)$ must have its values concentrated in a neighborhood of t. We say that it is localized in time. D. Gabor, was the first to propose a transform with the above properties.

Window function

A non-trivial function $g(x) \in L^2(\mathbb{R})$ is called a window function if $tg(t) \in L^2(\mathbb{R})$. An example of such a window function is the Haar function.

$$f(t) = \begin{cases} 1, & 0 \le t < 1/2 \\ -1, & 1/2 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Window function(Haar function)



Windowed Fourier transform

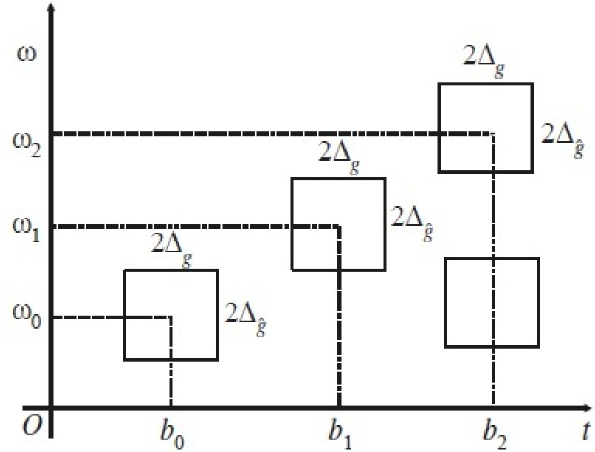
We define the Windowed Fourier transform of a function $f \in L^2(\mathbb{R})$ with respect to a windowed function g evaluated at a location (b, ω) in the time-frequency plane as

$$G_g f(b, \omega) = \int_{-\infty}^{\infty} f(t)\overline{g(t-b)}e^{-iwt}dt$$

Advantage over Fourier transform

Unlike the case of Fourier transform in which the function f must be known for the entire time axis before its spectral component at any single frequency can be computed,short-time Fourier transform(STFT) or windowed Fourier transform needs to know $f(t)$ only in the interval in which $g(t-b)$ is non zero.

Time-frequency window for STFT



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- The intuition behind this principle is simple: To measure frequencies we must observe the signal for some period of time.
- The more precision we need in the frequency measurements the larger the time interval we have to observe.

The windowed Fourier transform introduces a scale (the width of the window), and analyzes the signal from the point of view of that scale. If the signal has important frequency details outside of the scale, we will have problems in the signal analysis:

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- If the signal details are much smaller than the width of the window, we will have a problem similar to the one we faced with the Fourier transform: The details will be detected but the transform will not localize them.
- If the signal details are larger than the width of the window, they will not be detected properly.

To solve this problem when we analyze a signal using the windowed Fourier transform, we must define a transform which is independent of scale. This transform should not use a fixed scale, but a variable one.

Where doWavelets Fit?

In order to understand the role of the wavelets in the scenario of computational mathematics, even without understanding what a wavelet is, we must remember that our major concern is the description, representation, and reconstruction of functions. The different uses of wavelets in computationalmathematics, and in particular in computer graphics, are related with two facts:

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- Representation and reconstruction of functions;
- Multiresolution representation, a problem that consists in representing the graphics object in different resolutions.

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- The scale is defined by the width of the modulation function. Therefore we must use a modulation function which does not have a fixed width.Moreover the function must have good time localization.

Wavelet

A function $\psi \in L^2(\mathbb{R})$ is called a basic wavelet or mother wavelet if the following condition

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} < \infty$$

is satisfied where $\hat{\psi}(\xi)$ is the Fourier transform of $\psi(t)$. This condition is known as the *admissibility condition*.

Wavelets are mathematical functions which are generated by the dyadic dilations and integer shifts a single function called a mother wavelet and are defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \; a, b \in \mathbb{R}, \; a \neq 0$$

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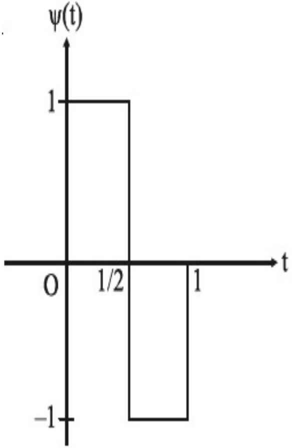
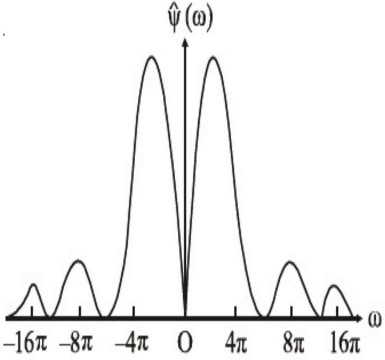
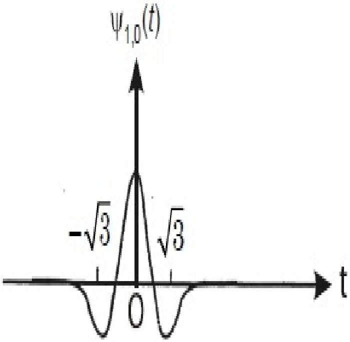
where a is called a scaling parameter which measures the degree of compression or scale, and b is a translation parameter which determines the time location of the wavelet.

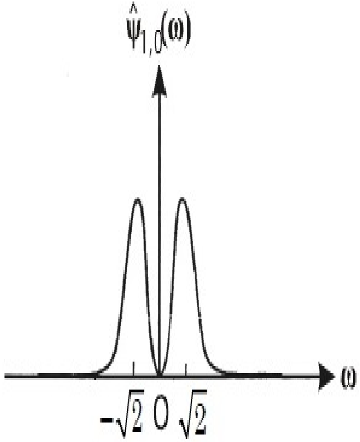
Haar Wavelet and its Fourier transform





Mexican hat Wavelet and its Fourier transform

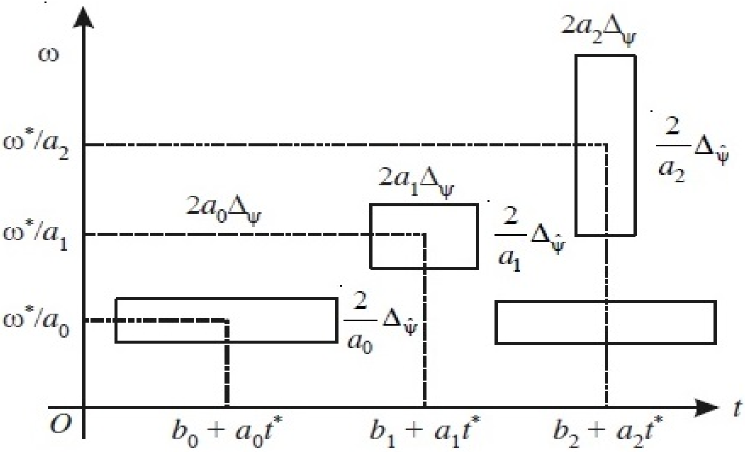




The continuous wavelet transform W_ψ of a function $f \in L^2(\mathbb{R})$ with respect to wavelet ψ is defined as

$$W_\psi f(a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(t)\overline{\psi_{a,b}(t)}dt,$$

Time-frequency window for continuous wavelet transform



Inverse wavelet transform

The inverse wavelet transform is given by

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_\psi f(a,b) \psi_{a,b}(t) \frac{dadb}{a^2},$$

where

$$C_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} < \infty$$

is known as *admissibility condition*.

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THANKS!!